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ON CONVECTIVE INSTABILITY OF AIR IN THE SNOW COVER

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This paper considers the system of equations describing the convective instability of air in the snow cover. We analyze the conditions under which the principle of "monotony" of normal perturbations is fulfilled and estimate the critical parameters of the main level of air equilibrium instability with allowance for the heat and mass exchange between the snow surface and the environment.

Equations of Thermal Convection of Air in the Snow Cover. Snow is a porous medium consisting of a rigid framework formed by randomly arranged ice crystals and pores filled with air that can pass through them both inwards and outwards. The temperature field in snow is inhomogeneous as a rule. When the vertical temperature gradient exceeds a certain threshold value, inside the snow macroscopic motion of air filling the pores in it appears. It is described by a system of hydrodynamical equations including the equations of motion, heat transfer, and continuity [1].

The equation of motion can be written in the form

$$\rho f \frac{d\mathbf{V}}{dt} = - f\nabla p + f\rho\mathbf{g} + \mathbf{F}\,, \tag{1}$$

where $\mathbf{F}$ is the friction force that arises when the air moves relative to the crystalline framework. Its value, assigned to the unit volume of the porous medium, according to the Darcy law, is

$$\mathbf{F} = \frac{f\mu}{\sigma}\mathbf{V}\,. \tag{2}$$

Equation (1), taking into account (2), can be written as

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\,\mathbf{V} = -\frac{1}{\rho}\nabla p + \mathbf{g} - \frac{\nu}{\sigma}\mathbf{V}\,. \tag{3}$$

The heat flux density $\mathbf{j}$ in the snow is determined by the formula

$$\mathbf{j} = -\lambda_{\mathrm{s}}\nabla\theta - L_{\mathrm{s}}D_{\mathrm{w}}\nabla\rho_{\mathrm{w}} + f\rho c_p\theta\mathbf{V} + L_{\mathrm{s}}f\rho_{\mathrm{w}}\mathbf{V}\,. \tag{4}$$

In so doing, the equation of heat transfer in the snow takes on the form

$$\frac{\partial}{\partial t}(\rho_{\mathrm{s}}c_{\mathrm{s}}\theta) + \mathrm{div}\,\mathbf{j} = 0\,. \tag{5}$$

The temperature dependence of the saturating vapor density, with the aid of the Clapeyron–Clausius equation, can be written in the form

$$\rho_{\mathrm{w}} = \rho_{\mathrm{w}0}\left(1 + \frac{L_{\mathrm{s}}\theta}{R_{\mathrm{w}}T_0^2}\right). \tag{6}$$

Taking into account equalities (4) and (6), we represent Eq. (5) as

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$$\frac{\partial\theta}{\partial t} + M\mathbf{V}\,\nabla\theta + M\theta\,\mathrm{div}\,\mathbf{V} + \frac{\rho_{w0}L_s}{\rho_s c_s}\,\mathrm{div}\,\mathbf{V} = \chi_s\,\Delta\theta\,, \tag{7}$$

where

$$\chi_s = \frac{\lambda_s}{\rho_s c_s}\left(1 + \frac{L_s^2\rho_{w0}\lambda_s}{R_w D_s T_0^2}\right) \tag{8}$$

is the effective thermal diffusivity of the snow, and

$$M = \frac{f\rho c_p}{\rho_s c_s}\left(1 + \frac{L_s^2\,\rho_{w0}}{\rho R_w c_p T_0^2}\right). \tag{9}$$

The second terms between brackets in equalities (8) and (9) are equal to 0.1–0.2 and 0.6–0.7, respectively.

For the air piercing the snow framework, the continuity equation

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\mathbf{V}) = 0\,. \tag{10}$$

is met.

The system of equations (3), (7), and (10) describing the air motion in the snow should be complemented by the equation of state

$$\rho = \frac{fp}{RT}\,. \tag{11}$$

We begin the derivation of the equation of free convection of air in the snow by simplifying formula (11).

We represent the temperature T and the pressure p in the form $T = \langle T\rangle + T'$, $p = \langle p\rangle + p'$, where $\langle T\rangle$ and $\langle p\rangle$ are certain constant mean values taken as references and the components T' and p' are assumed to be small. Then, restricting ourselves to the terms linear in T' and p', we can write

$$\rho = \rho_0 + \left(\frac{\partial\rho}{\partial T}\right)_p T' + \left(\frac{\partial\rho}{\partial p}\right)_T p' = \rho_0\,(1 - f\beta T' + \alpha p' f)\,,$$

where $\alpha = \frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial p}\right)_T = \frac{1}{p}$ and $\beta = -\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial p}\right)_p = \frac{1}{T}$ are the coefficients of isothermal compression and thermal expansions of the air.

Besides assuming small T' and p', we also assume that $f\alpha p' = \left|f\beta T'\right|$. We show that this condition is always met in snow.

If l is the characteristic vertical size, then the hydrostatic differential pressure is of the order of $\rho_0 gl$, where $g = |\mathbf{g}|$, and the condition $\alpha p' << |\beta T'|$ takes on the form

$$\left|\frac{l\rho g\alpha}{\rho T}\right| = \left|\frac{gl}{RT}\right| << 1\,. \tag{12}$$

Replacing in the obtained inequality g/R by $g/c\rho$, we have $\gamma_a << \gamma_s$, where $\gamma_a = 10^{-2}$ deg/m. Under real conditions, $\gamma_a << \gamma_s$.

Taking into account inequality (12), for the air density in the snow we obtain

$$\rho = \rho_0\,(1 - f\,\beta T')\,,$$

and the condition of smallness of the relative changes in the air density allows us to write the continuity equation as

$$\operatorname{div} \mathbf{V} = 0 \,. \tag{13}$$

Thus, the system of equations of thermal convection in the snow in the Boussinesq approximation can be given in the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\,\mathbf{V} = -\frac{1}{\rho_0}\nabla p - \frac{\nu}{\sigma}\mathbf{V} + f\beta g\theta\mathbf{e}_z\,, \quad \frac{\partial \theta}{\partial t} + M\,\mathbf{V}\,\nabla\theta = \chi_s\,\Delta\theta\,, \quad \operatorname{div}\mathbf{V} = 0\,. \tag{14}$$

For short, the prime in p' was omitted and T' was replaced by θ.

The obtained system of equations (14) differs from the analogous system of equations of thermal convection of an incompressible fluid in that the motion equation, instead of the term $\nu\Delta\mathbf{V}$, has the term $-\frac{\nu}{\delta}\mathbf{V}$, and in the heat-conductivity equation the term $\mathbf{V}\nabla\theta$ has acquired a factor M of the order of 1.6–1.7.

If the snow mass is contained in a cavity surrounded by a solid homogeneous heat-conducting mass, it is necessary to add to the system of equations (14) the heat-conductivity equation determining the temperature field in the mass:

$$\frac{\partial \theta_m}{\partial t} = \chi_m \Delta\theta_m \,. \tag{15}$$

It is necessary to add the boundary conditions to the system of equations (14)–(15): at the snow–solid-mass interface the velocity goes to zero, $\mathbf{V} = 0$, and the temperature and the normal component of the thermal flow are continuous: $\theta = \theta_m$, $\lambda_s \frac{\partial \theta}{\partial n} = \lambda_m \frac{\partial \theta_m}{\partial n}$.

We now consider the small nonstationary perturbations of equilibrium. Then in the equations that are obtained by substituting ($\mathbf{V}_1$, $\theta_0 + \theta_1$, $p_0 + p_1$, $\theta_{m0} + \theta_1$) into (14)–(15) the terms quadratic in perturbed members can be neglected, and, as a result, we obtain

$$\frac{\partial \mathbf{V}_1}{\partial t} = -\frac{1}{\rho_0}\nabla p_1 - \frac{\nu}{\sigma}\mathbf{V}_1 + f\beta g\theta_1\mathbf{e}_z\,, \quad \frac{\partial \theta_1}{\partial t} + M\,\mathbf{V}_1\,\nabla\theta_0 = \chi_s\,\Delta\theta_1\,, \quad \operatorname{div}\mathbf{V}_1 = 0\,, \quad \frac{\partial \theta_{m1}}{\partial t} = \chi_m\Delta\theta_1\,. \tag{16}$$

Let us write the system of equations (16) in dimensionless form. To this end, as a unit length, we choose the characteristic linear size of the cavity L, as a unit time — σ/ν, and a velocity unit — $\chi_s/(ML)$, and we measure the pressure and temperature perturbations in units $\rho_0\nu\chi_s/\sigma$ and $\gamma_0 L$, respectively. Then the representation of the linear system of equations of free convection of air in the snow will take on the form

$$\frac{\partial \mathbf{V}}{\partial t} = -M\nabla p - \mathbf{V} + \mathrm{Ra}'\,\theta\mathbf{e}_z\,, \quad \mathrm{Pr}'\frac{\partial \theta}{\partial t} - (\mathbf{V}\,\mathbf{e}_z) = \Delta\theta\,, \quad \operatorname{div}\mathbf{V} = 0\,, \quad \mathrm{Pr}'\,\overline{\chi}\,\frac{\partial \theta_m}{\partial t} = \Delta\theta_m\,, \tag{17}$$

where $\mathrm{Ra}' = f\beta g M\sigma\gamma_0 L^2/(\chi_s\nu)$ is a dimensionless parameter, which is an analog of the Rayleigh number $\mathrm{Ra} = \beta g L^4\gamma_0/(\nu\chi)$ for an incompressible liquid; $\mathrm{Pr}' = \nu L^2/(\chi_s\sigma)$ is an analog of the Prandtl numbers $\mathrm{Pr} = \nu/\chi$; $\overline{\chi} = \chi_s/\chi_m$.

Now in the system of equations (17) $\mathbf{V}$, p, θ, and θ_m are the dimensionless perturbations and all derivatives are taken with respect to the linear coordinates and time. Thus, the small perturbations satisfy the system of linear homogeneous equations with constant coefficients.

Horizontal Homogeneous Snow Cover. We shall further consider a snow cover limited by horizontal planes $z = 0$ and $z = H$ and lying on the top of a layer of frozen soil of thickness H_1 (see Fig. 1). The behavior of the small perturbations of air in the snow cover is described by the system of equations (17). We exclude from this system the pressure p and the horizontal components of the velocity, V_x and V_y. To do this, we apply to this equation the rotrot operation and project the resulting vector equation on the z-axis. As a result, we have a system of three equations for the vertical velocity component V_z and for the perturbations of temperatures θ and θ_m:

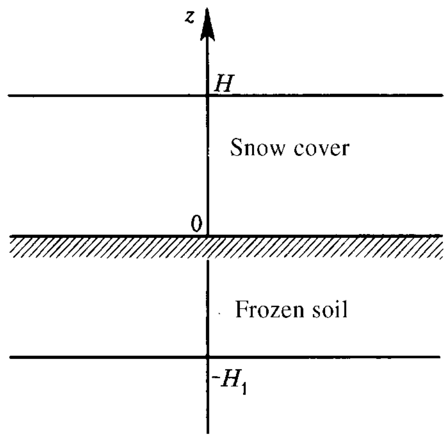

Fig. 1. Scheme of the two-layer medium

$$\frac{\partial}{\partial t}(\Delta V_z) = \mathrm{Ra}' \, \Delta_1 \theta - \Delta V_z \, , \tag{18}$$

$$\mathrm{Pr}' \frac{\partial \theta}{\partial t} = \Delta \theta + V_z \, , \tag{19}$$

$$\mathrm{Pr}' \bar{\chi} \frac{\partial \theta_{\mathrm{m}}}{\partial t} = \Delta \theta_{\mathrm{m}} \, , \tag{20}$$

where $\Delta_1 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is a plane Laplace operator.

We now formulate the system of boundary conditions.

Since frozen soil forms a solid mass, which we assume to be impermeable to air, the velocity of air on its surface goes to zero:

$$V_z(0) = 0 \, . \tag{21}$$

If the snow surface is covered with a thin layer of ice, the vertical air velocity component on it is also equal to zero:

$$V_z(1) = 0 \, . \tag{22}$$

But if the snow surface is permeable to air, then, as follows from the continuity equation, the following condition should be met:

$$V_z'(1) = 0 \, , \tag{23}$$

where the prime denotes differentiation with respect to *z*.

Turning to the formulation of the boundary conditions for the temperature, note that under real conditions the freezing of the soil is so slow that the temperatures in the snow cover and in the frozen soil are close to equilibrium. In this case, the equation for the thermal balance on the snow surface, taking into account the heat- and mass-exchange between this surface and the environment, can be written in the form

$$-\lambda_{\mathrm{s}} \frac{d\theta}{dz}\bigg|_{z=H} = \alpha' (\theta_{\mathrm{s}} - \theta_\infty) + \beta' L_{\mathrm{s}} (\rho_{\mathrm{w}} - \varepsilon \rho_{\mathrm{w}\infty}) \, , \tag{24}$$

where

$$\rho_{\mathrm{w}} \approx \rho_{\mathrm{w0}} \left(1 + \frac{L_{\mathrm{s}}\theta_{\mathrm{s}}}{R_{\mathrm{w}}T_0^2}\right); \quad \rho_{\mathrm{w}\infty} \approx \rho_{\mathrm{w0}} \left[1 + \frac{L_{\mathrm{s}}\theta_{\mathrm{s}}}{R_{\mathrm{w}}T_0} - \frac{L_{\mathrm{s}}(\theta_{\mathrm{s}} - \theta_\infty)}{R_{\mathrm{w}}T_\infty^2}\right].$$

Equality (24) is rigorous only in the case where the snow surface is covered with a thin layer of ice and, consequently, is closed to air. In the case of an open surface where the air can freely go from the snow outwards, it is necessary to add the term $\rho c_p \varepsilon \theta_{\mathrm{s}} V_z$ to the right-hand side of this equality. As estimates show, however, this term is small compared to the other ones, and we shall further neglect it.

Let us assume that the snow-surface temperature θ_{s} receives a small perturbation θ_1, and the saturating vapor density at the same temperature also receives a small perturbation $\rho_{\mathrm{w1}} \approx \rho_{\mathrm{w0}} L_{\mathrm{s}} \theta_1 / (R_{\mathrm{w}} T_0^2)$. By virtue of equality (24) this increment satisfies the equation

$$-\lambda_{\mathrm{s}} \left.\frac{d\theta_1}{dz}\right|_{z=H} = \left(\alpha' + \beta' \frac{\rho_{\mathrm{w0}} L_{\mathrm{s}}^2}{R_{\mathrm{w}} T_0^2}\right) \theta_1 .$$

On going to dimensionless quantities, from this we obtain

$$\theta'(1) + h\theta(1) = 0 , \tag{25}$$

where

$$h = \frac{\alpha' H}{\lambda_{\mathrm{s}}} \left(1 + \frac{\beta'}{\alpha'} \frac{\rho_{\mathrm{w0}} L_{\mathrm{s}}^2}{R_{\mathrm{w}} T_0^2}\right). \tag{26}$$

As is known [3], the heat-exchange coefficient $\alpha' = \rho c_p U$, where U is the characteristic velocity of the air flow at a certain distance l' from the snow surface, and the mass-exchange coefficient $\beta' = l'/D_{\mathrm{t}}$, where $D_{\mathrm{t}} \approx l'U$. Then equality (26) takes on the form

$$h = \frac{\rho c_p H U}{\rho_{\mathrm{s}} c_{\mathrm{s}} \chi_{\mathrm{s}}} \left(1 + \frac{\rho_{\mathrm{w0}} L_{\mathrm{s}}^2}{\rho c_p R_{\mathrm{w}} T_0^2}\right). \tag{27}$$

At the snow–frozen soil interface the conjugation conditions

$$\theta(0) = \theta_{\mathrm{m}}(0) , \quad \theta'_{\mathrm{m}}(0) = \bar{\chi}\, \theta'(0) , \tag{28}$$

where $\bar{\chi} = \lambda_{\mathrm{s}}/\lambda_{\mathrm{m}}$ should be met; at this frozen soil boundary the temperature is fixed and equal to zero; consequently:

$$\theta_{\mathrm{m}} \left(-\frac{H_1}{H}\right) = 0 . \tag{29}$$

Thus, the small perturbations of air equilibrium in the snow cover satisfy the system of linear homogeneous equations in partial derivatives with constant coefficients (18)–(20), which should satisfy the homogeneous boundary conditions (21)–(23), (25) and (28), and (29). Such systems, as is known, have partial solutions describing the so-called "normal" perturbations that are exponentially dependent on time and periodic in the (x, y) plane:

$$V_z(x, y, z, t) = V(z) \exp[-\lambda t + i(k_1 x + k_2 y)] ,$$

$$\theta(x, y, z, t) = \Theta(z) \exp[-\lambda t + i(k_1 x + k_2 y)] ,$$

$$\theta_m(x, y, z, t) = \Theta_m(z) \exp[-\lambda t + i(k_1 x + k_2 y)], \tag{30}$$

where $V(z)$, $\Theta(z)$, and $\Theta_m(z)$ are the perturbation amplitudes.

Substituting (30) into (18)–(20), we obtain

$$\lambda(V'' - k^2 V) = (V'' - k^2 V) + k^2 \mathrm{Ra}' \Theta, \tag{31}$$

$$-\lambda \mathrm{Pr}' \Theta = (\Theta'' - k^2 \Theta) + V, \tag{32}$$

$$-\lambda \mathrm{Pr}' \bar{\chi} \Theta = (\Theta_m'' - k^2 \Theta_m). \tag{33}$$

The amplitude equations (31)–(32) together with the homogeneous conditions (21)–(23) and (28), (29) form the spectral problem determining the characteristic perturbations and their decrements for the snow cover.

"Monotony Principle" of Perturbations for the Snow Cover. Let us establish one important property of the spectrum following from Eqs. (31)–(33) and the boundary conditions formulated above.

As in investigating the spectrum of small perturbations of incompressible liquid equilibrium [2], along with the solutions (λ, V, Θ, Θ_m) of Eqs. (31)–(33), we consider the complex-conjugate solutions (λ^*, V^*, Θ^*, Θ_m^*) (asterisks mark the complex-conjugate quantities). Multiply Eq. (31) by V^* and integrate from 0 to 1:

$$\lambda \int_0^1 (V'' - k^2 V) V^* dz = \int_0^1 (V'' - k^2 V) V^* dz + k^2 \mathrm{Ra}' \int_0^1 V^* \Theta dz. \tag{34}$$

Taking into account that the integral

$$\int_0^1 (V'' - k^2 V) V^* dz = V' V^* \big|_0^1 - \int_0^1 [|V'|^2 + k^2 |V|^2] dz,$$

equality (34), taking into account the boundary conditions (31)–(33), can be written in the form

$$(\lambda - 1) \int_0^1 [|V'|^2 + k^2 |V|^2] dz = -k^2 \mathrm{Ra}' \int_0^1 \Theta V^* dz. \tag{35}$$

Passing in (35) to the complex-conjugate values, we obtain

$$(\lambda^* - 1) \int_0^1 [|V'|^2 + k^2 |V|^2] dz = -k^2 \mathrm{Ra}' \int_0^1 \Theta^* V dz. \tag{36}$$

Subtraction from (35) of equality (36) yields

$$(\lambda - \lambda^*) \int_0^1 [|V'|^2 + k^2 |V|^2] dz = -k^2 \mathrm{Ra}' \int_0^1 \Theta^* V dz. \tag{37}$$

Performing the same operations with Eqs. (32) and (33) and taking into account the boundary conditions (25), (28), and (29), we have

$$(\lambda - \lambda^*)\,\mathrm{Pr}'\int_0^1 |\Theta|^2\,dz = \int_0^1 (V^*\Theta - V\Theta^*)\,dz + \Theta(0)\,\Theta^{*\prime}(0) - \Theta'(0)\,\Theta^*(0)\,, \tag{38}$$

$$(\lambda - \lambda^*)\,\mathrm{Pr}'\int_{-H_1/H}^{0} |\Theta_{\mathrm{m}}|^2\,dz = -\,[\Theta(0)\,\Theta^{*\prime}(0) - \Theta'(0)\,\Theta^*(0)]\,. \tag{39}$$

Adding together (38) and (39), we obtain

$$(\lambda - \lambda^*)\,\mathrm{Pr}'\left[\int_0^1 |\Theta|^2\,dz + \int_{-H_1/H}^{0} |\Theta_{\mathrm{m}}|^2\,dz\right] = \int_0^1 (V^*\Theta - V\Theta^*)\,dz\,. \tag{40}$$

Now multiply equality (40) by $k^2\,\mathrm{Ra}'$ and add to (37); in so doing, we have

$$(\lambda - \lambda^*)\left[\int_0^1 \Big[\,|V'|^2 + k^2\,|V|^2 + k^2\,\mathrm{Pr}'\,\mathrm{Ra}'\,|\Theta|^2\Big]dz + k^2\,\mathrm{Pr}'\,\mathrm{Ra}'\int_{-H_1/H}^{0} |\Theta_{\mathrm{m}}|^2\,dz\right] = 0\,. \tag{41}$$

At $\mathrm{Ra}' > 0$ (heating from below) the expression between square brackets of equality (41) is always positive. Consequently, the equality $\lambda = \lambda^*$ should take place.

Thus, at positive values of the Rayleigh number ($\mathrm{Ra}' > 0$) the decrements of normal perturbations are real and all normal perturbations either decay or increase monotonically (monotony principle of perturbations) [2].

As in the case of an incompressible liquid equilibrium, in the case where the snow cover is heated from below, there is a sequence of Rayleigh numbers, upon reaching which the equilibrium of the corresponding air perturbation in the snow becomes unstable. When the above-formulated boundary conditions for V and Θ, perturbations with $\lambda > 0$ decay with time and the equilibrium is stable, whereas at $\lambda < 0$ they increase and the equilibrium is unstable; the value of $\lambda = 0$ gives a neutral perturbation separating stable and unstable perturbations. Assuming in the amplitude equations (31)–(33) $\lambda = 0$, we thus obtain the boundary-value problem for the neutral perturbations

$$(V'' - k^2 V) + k^2\,\mathrm{Ra}'\,\Theta = 0\,, \quad (\Theta'' - k^2\Theta) + V = 0\,, \quad \Theta_{\mathrm{m}}'' - k^2\Theta_{\mathrm{m}} = 0 \tag{42}$$

with the homogeneous boundary conditions (21)–(23), (25), (28), and (29). In Eq. (42), double primes denote double differentiation with respect to z. The eigennumbers of this problem are the critical Rayleigh numbers Ra' and the eigenfunctions are the amplitudes of critical perturbations.

In [1], the critical parameters of the main level of air stability in the snow in the case where the velocity satisfies the boundary conditions (21)–(23) and the temperature satisfies the conditions $\Theta(0) = \Theta(1) = 0$ have been determined.

Below, we consider a more general case where the temperature satisfies the boundary-value condition (25).

Determination of the Critical Parameters of the Main Level of Air Instability in the Snow Cover. Assume that the temperature at the snow–frozen soil interface is fixed and the snow surface is permeable to air and exchanges heat with the environment by law (24). Then the determination of the critical parameters of the main level of air instability is reduced to the solution of the first two equations of system (42) under the following boundary conditions:

$$V(0) = 0\,, \quad V'(0) = 0\,, \quad \Theta(0) = \Theta(1) = 0\,, \quad \Theta'(1) + h\,\Theta(1) = 0\,.$$

We solve the problem approximately. To do this, we make use of the second, modified variant of the Bubnov–Galerkin method [2]. As the basis function, we take the function

$$V(z) = \sin\frac{\pi z}{2}\,, \tag{43}$$

satisfying the boundary conditions (21)–(23).

Substituting (43) into the second equation of system (42) and solving the obtained equation, we have

$$\Theta = \frac{1}{k^2 + \dfrac{\pi^2}{4}}\left(\sin\frac{\pi z}{2} - \frac{\sinh kz}{\dfrac{k}{h}\cosh k + \sinh k}\right). \tag{44}$$

Substituting (43) and (44) into the first equation of system (42), we obtain

$$\left(k^2 + \frac{\pi^2}{4}\right)^2 \sin\frac{\pi z}{2} = k^2\,\mathrm{Ra}'\left(\sin\frac{\pi z}{2} - \frac{\sinh kz}{\dfrac{k}{h}\cosh k + \sinh k}\right). \tag{45}$$

Multiply either side of Eq. (45) by $V_z = \sin\dfrac{\pi z}{2}$ and integrate from 0 to 1. Taking into account that the integral

$$\int_0^1 \sinh kz\,\sin\frac{\pi z}{2}\,dz \equiv \frac{k\cosh k}{k^2 + \dfrac{\pi^2}{4}}\,,$$

we have

$$\mathrm{Ra}' = \frac{\left(k^2 + \dfrac{\pi^2}{4}\right)^3}{k^2\left(k^2 + \dfrac{\pi^2}{4} - \dfrac{2k}{\dfrac{k}{h} + \tanh k}\right)}\,. \tag{46}$$

We find the critical value of the wave number at which the Ra' number takes on the minimum value from the condition $\partial\mathrm{Ra}'/dk = 0$, which leads to the following transcendental equation:

$$3k^2\left(k^2 + \frac{\pi^2}{4} - \frac{2k}{\dfrac{k}{h} + \tanh k}\right) = \left(k^2 + \frac{\pi^2}{4}\right)\left[2k^2 + \frac{\pi^2}{4} - \frac{2k}{\dfrac{k}{h} + \tanh k} - \frac{k\tanh k - \dfrac{k^2}{\cosh^2 k}}{\left(\dfrac{k}{h} + \tanh k\right)^2}\right]. \tag{47}$$

Equation (47) contains only a single parameter h depending on the conditions of heat- and mass-exchange between the snow surface and the environment. We assume the following characteristic features of the parameters entering into equality (27): $\rho_s = 300\ \mathrm{kg/m^3}$, $c_p = 1$ kJ/(kg·deg), $c_s = 1.38$ kJ/(kg·deg), $\chi_s = 4\cdot 10^{-8}\ \mathrm{m^2/sec}$, $L_s = 2850$ kJ/kg, $R_w = 4.6$ kJ/(kg·deg), $\rho = 1.29\ \mathrm{kg/m^3}$, $\rho_{w0} = 6.2\cdot 10^{-4}\ \mathrm{kg/m^3}$, $H = 0.5$ m, $U = 0.01$ m/sec, and then $h \approx 40$.

The numerical solution of Eq. (47) yields the following values for the critical parameters of the main level of instability: $k_{\min} = 2.5$ and $\mathrm{Ra}'_{\min} = 24.8$ at $h = 20$; $k_{\min} = 2.3$ and $\mathrm{Ra}'_{\min} = 28.9$ at $h = 40$; $k_{\min} = 2.3$ and $\mathrm{Ra}'_{\min} = 32$ at $h \to \infty$.

We now consider the case where the snow surface is impermeable to air: $V(0) = V(1) = 0$. As the basis function, we take

$$V(z) = \sin \pi z\,. \tag{48}$$

In this case,

$$\Theta = \frac{1}{k^2 + \pi^2}\left(\sin \pi z + \frac{\pi \sinh kz}{k \cosh k + h \sinh k}\right). \tag{49}$$

As above, substitute (48) and (49) into the first equation of system (42) and integrate from 0 to 1. As a result, we have

$$\mathrm{Ra}' = \frac{(k^2 + \pi^2)^3}{k^2\left(k^2 + \pi^2 + \dfrac{2\pi^2 \tanh k}{k + h \tanh k}\right)}.$$

When $h \to \infty$ we obtain from this the known results [2]: $k_{\min} = \pi$, $\mathrm{Ra}'_{\min} = 4\pi^2$. At $h \neq \infty$ the value of $k_{\min}$ differs but slightly from π and $\mathrm{Ra}'_{\min} \approx 4\pi^2\left(1 - \dfrac{1}{\pi + h}\right)$.

Thus, in the general case, with increasing h the critical values of $k_{\min}$ and $\mathrm{Ra}'_{\min}$ tend to the limiting values that are obtained at a fixed value of the snow-surface temperature. In this case, as would be expected, a "weakening" of the boundary condition for the air velocity leads to a decrease in the critical value of $\mathrm{Ra}'_{\min}$ and an increase in the critical wavelength.

The critical temperature gradient in the snow cover at which loss of equilibrium stability begins is determined by the formula

$$\gamma_{\mathrm{cr}} = \frac{\nu \chi_{\mathrm{s}} \mathrm{Ra}'_{\min}}{\beta f g M H^2 \sigma}. \tag{50}$$

At values of the permeability coefficient $\sigma \approx 10^{-8}\ \mathrm{m}^2$, of the density of $\rho = 0.3\ \mathrm{kg/m}^3$, and $h = 40$ that are characteristic of snow, by formula (5) $\gamma_{\mathrm{cr}} \approx 0.16$ deg/m is obtained, i.e., this value exceeds the dry-adiabatic temperature gradient by a factor of 16.

Thus, the small temperature gradients in the snow cover caused by a change in the ambient temperature lead to the appearance of convection in the bulk of the snow. From this it follows that the chief factor determining the processes of heat and mass transfer in the snow cover is the convective motion of the air filling the pores in the snow, and the molecular heat conductivity and diffusion play only a secondary role.

In conclusion, note that the above results can be extended to a cultivated layer of soil as well as to any free-flowing material containing air inclusions. In particular, formula (50), when properly corrected, can be used to determine the possibility of the appearance of convective motion of air in the masses of grain, potatoes, and other vegetables in granaries and vegetable store-houses by the value of the temperature gradient in them.

NOTATION

F, friction force; ρ, air density; f, porosity factor (ratio of the pore volume to the total volume of the snow); **V**, mass velocity of air in the snow; p, air pressure; **g**, acceleration of gravity; μ, air viscosity; σ, permeability coefficient of snow; t, time; z, vertical coordinate; ν, kinematic viscosity of air; **j**, heat flux; λ_{s}, heat-conductivity coefficient of snow; ∇, gradient operator; θ, temperature of snow, $^{\mathrm{o}}$C; L_{s}, specific heat of evaporation of snow (ice); D_{w}, diffusion coefficient of water vapor in air; ρ_{w}, saturating density of vapor; c_p, heat capacity of air at a constant pressure; ρ_{s}, snow density; c_{s}, heat capacity of snow; ρ_{w0}, saturating density of vapor at 0^{o}C; R_{w}, gas constant of water vapor; $T_0 = 273$ K; χ_{s}, effective thermal diffusivity of snow; R, gas constant of air; T, absolute temperature; T', temperature perturbation; p', pressure perturbation; $\langle T\rangle$, mean value of the temperature; $\langle p\rangle$, mean value of the pressure; ρ_0, mean air density; α, isothermal compressibility coefficient of air; β, coefficient of thermal expansion of air; l, characteristic linear scale of vertical; γ_{a}, dry-adiabatic temperature gradient; γ_{s}, temperature gradient in the snow cover; $\mathbf{e}_z$, unit vector along the z-axis; θ_m, mass (of frozen soil) temperature, ^oC; χ_m, thermal diffusivity of the mass; Δ, Laplace operator; λ_m, heat-conductivity coefficient of the mass; n, normal to the snow–solid mass interface; θ_1, small nonstationary perturbations of temperature; p_1, small nonstationary perturbations of pressure; θ_0, equilibrium temperature in the snow; θ_{m0}, temperature in the mass; $\mathbf{V}_1$, nonstationary perturbations of the air velocity in the snow; θ_{m1}, nonstationary temperature perturbations in the mass; L, linear size of the cavity; γ_0, equilibrium temperature gradient in the snow; Ra, Rayleigh number for an incompressible liquid; Pr, Prandtl number for an incompressible liquid; Ra', Rayleigh number for snow; Pr', Prandtl number for snow; H, thickness of the snow cover; α', coefficient of heat exchange between the snow surface and the environment; β', coefficient of mass exchange between the snow surface and the environment; θ_s, temperature of the snow surface; $\rho_{w\infty}$, saturating vapor density at a distance from the snow surface; ε, relative humidity of air; T_∞, absolute temperature of air; ρ_{w1}, small increment of the water vapor density in air; U, characteristic velocity of air over the snow surface; l', characteristic distance from the snow surface; D_t, turbulent heat diffusivity of air; H_1, depth of the frozen soil layer; λ, decrement; k_1 and k_2, real wave numbers; $\Theta(z)$, amplitude of temperature perturbations in the snow; $\Theta_m(z)$, amplitude of temperature perturbations in the mass; γ_{cr}, critical temperature gradient in the snow; k, wave number; h, dimensionless quantity. Subscripts: cr, critical; a, adiabatic; w, water vapor; s, snow; t, turbulent; m, mass (frozen soil).

REFERENCES

1. M. K. Zhekamukhov and L. Z. Shukhova, *Prikl. Mekh. Tekh. Fiz.*, **40**, No. 6, 54–59 (1999).
2. G. Z. Gershuni and E. M. Zhukhovistkii, *Convective Stability of Incompressible Fluid* [in Russian], Moscow (1972).
3. L. D. Landau and E. M. Lifshits, *Hydrodynamics* [in Russian], Moscow (1986).